Abstract

This paper will discuss information theorist Gregory Chaitin's claim that randomness is a characteristic of arithmetic. The discussion strongly suggests the need for a rethinking of metaphysics. I will follow the problem of algebraic countability from Hiblert, Gödel and Turing and end with Gregory Chaitin's demonstration of randomness in arithmetic. The significant claim that randomness is a characteristic of attempting to answer some important but very unusual questions in the logic of arithmetic will be examined in light of the metaphysical implications. If one can find issue in the system of arithmetic then this means that the same issues will permeate in other similar systems. Chaitin claims that a primacy of information is necessary for an appropriate understanding of the issue of randomness in arithmetic which leads me to the speculation that a new type of information process metaphysics is the key to bridging process philosophy with a modern mathematical-scientific understanding of the world.

THE INFORMATION PROCESS RELATIONSHIP IN MATHEMATICAL METAPHYSICS: THE PROBLEM OF RANDOMNESS IN THE LOGIC OF ARITHMETIC

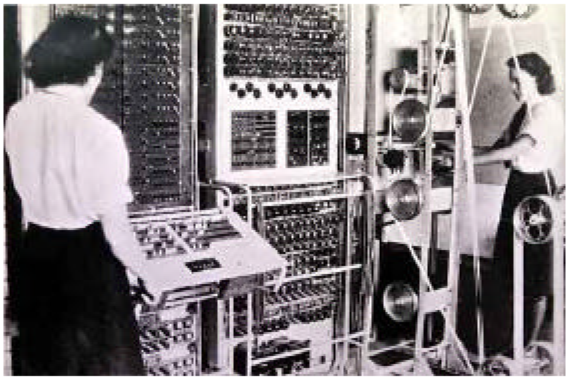
Abstract and paper accepted, “The Information Process Relationship In Mathematical Metaphysics: The Problem Of Randomness In The Logic Of Arithmetic,” 6th International Whitehead Conference, Salzburg, Austria, 3-6 July, 2006.

Marc J.V. Corbeil
BA Mathematics and Philosophy, Concordia University, Montreal
MA Philosophy University of North Texas
MTM Concordia University, Montreal

Doctoral Candidate
Cardiff University, University of Wales
Cardiff, UK

Faculty
Higher Colleges of Technology
PO box 17258
Al Ain, Abu Dhabi
United Arab Emirates
marc@mcorbeil.com

THE PROCESS INFORMATION RELATIONSHIP IN MATHEMATICAL METAPHYSICS: THE PROBLEM OF RANDOMNESS IN THE LOGIC OF ARITHMETIC



The Colossus, electronic digital computer, UK 1943.[1]

Hilbert's Problem

> I should say first of all, this: that it shall be possible to establish the correctness of the solution by means of a finite number of steps based upon a finite number of hypothesis which are implied in the statement of the problem and which must always be exactly formulated. This requirement of logical deduction by mean of a finite number of process is simply the requirement of rigor in reasoning. David Hilbert, 1900[2]

In Paris, 1900, David Hilbert, addressing the International Congress of Mathematicians, outlined 23 problems that remained to be adequately addressed by mathematicians. He threw down the gauntlet to the congress and urged them to fix mathematics truly and finally. Within the text of the second problem, "The Compatibility of the Arithmetical Axioms", Hilbert stated the common faith of mathematicians and scientists at the beginning of the twentieth century; A faith that has not, to a large extent, diminished in one hundred years since his address.

> I am convinced that it must be possible to find a direct proof for the compatibility of the arithmetical axioms, by means of a careful study and suitable modification of the known methods of reasoning in the theory of irrational numbers.[3]

According to Hilbert, a systematic search for meaningful statements in mathematics or well formed formulae and their proof in mathematics is a pursuit that is both possible and realistic. "The first step in the construction of an absolute proof [of mathematical statements], as Hilbert conceived the matter, is the *complete formalization* of a deduction system."[4] Recall that in a study of any formal axiomatic system, arithmetic for example, one is first concerned with logical consistency and completeness of the axiomatic system. Consistent means that a well formed formula and its negation both cannot be proved from the system's axioms. Thus, no contradictory statements within the formal system are possible, i.e. one cannot generate the falsity or F_o. Why is this important? The existence of F_o would mean that any statement, any well formed formula, could be proven; For any well formed formula P, F_o ✍ P is always true.[5] Completeness is also important. Completeness means that any well formed formula in the logic can either be proved or disproved, and recall that we call a proved a formula a theorem. In the logic of arithmetic, the logic that Hilbert was most interested in, this means any arithmetic theorem can be proved or disproved based on the axioms of arithmetic using the formal rules of inference of the system. Consequently, Hilbert's problem can be stated simply as a search for all possible well formed formulae and checking them to see if they are provable. The properties of completeness and consistency are verified along the way.

David Hilbert Was Not a Twit[6]

In his paper *Randomness in Arithmetic and the Decline and Fall of Reductionism in Pure Mathematics,* Gregory Chaitin suggests that we can formally expresses Hilbert's search as a simple algorithm that can be processed by a computer program. LOM (1-28) Such a program, an algorithmic program, would first generate all possible proofs of well formed formulae. To check for consistency means one could not prove both A and not A. So run through the algorithm that generates all possible proofs based on the axiom set and check for the occurrence of both A and not A. Since the program automatically evolved formulae that have proofs, completeness is not an issue. Either A or not A is a provable generated well formed formula. Generating all possible proofs seems to be a rather unreasonable thing to do since the universe of all possible proofs for an very simple arithmetic is rather large. In the early 1900s, this approach would have been implausible considering the tools available. Considering that Hilbert had the proving of the whole of mathematics in mind with the qualifying requirement that one accept only very formal demonstrations, this approach was simply not an option. Only a few mathematicians ever attempted processes or calculations at this scale and the results were not completely understood even then.[7] Hence the algorithmic approach was not the one first attempted in order to achieve Hilbert's requirements and purpose.

Bertrand Russell and Alfred North Whitehead would provide the tools for the attempt to formalize arithmetic. Through Russell and Whitehead, the goal of Hilbert's problem progressed into an effort to show that mathematics is reducible to logic.[8] Start with the axioms of arithmetic and generate, through proof, all of mathematical theory. This is only slightly less work than generating and checking all possible proofs but at least it became a problem of progressive proof rather than of mere computation. Only valid theorems would be demonstrated, and all the invalid ones would be thus avoided. Using a particular set of axioms of arithmetic that seemed promising, commonly labeled as Peano's axioms of arithmetic, Russell and Whitehead proceeded to construct set theory and number theory. The result is the monumental (but incomplete) three-volume tome *Principia Mathematica*[9] spanning thousands of pages of small print of axioms followed by proof and theorems. Never is doubt was the assumption that such a system would necessarily be consistent and that one would, in due course, prove everything known in mathematics, and perhaps much more. It was a smug, stiff and arrogant self-assurance set fully in the tradition of

Laplace's assumption. Not only was the idea dead wrong, but it lead to a proof of an idea far more dangerous than mere positivism.

Gödel's Incompleteness Theorem

Thirty-one years after Hilbert's speech in Paris Russell and Whitehead are found desperately working on the fourth volume of Principia Mathematica. Then, the paper *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*[10] appeared out of the school of logical positivism at the University of Vienna. In this paper Kurt Gödel demonstrated conclusively and without any doubt that the completeness of the axioms of arithmetic can never be proved! Hilbert's program is suddenly wrecked by an unsuspected and unusual theorem. The proof is a complicated set of self-references and the complexity of the proof requires painstaking work in which few, mathematicians or not, are willing to subject themselves. None the less, the result would shake the very foundations of mathematics and its impact is still being felt in the academics of mathematic today.

Gödel's Incompleteness Theorem: If you assume a system is consistent, then it must be incomplete.

A simple theorem but is meant that Hilbert's conviction was wrong, no search program can be consistent and complete. You can neither prove all the possible valid theorems without assuming a truth not proved. The kings of logic understood the result clearly: Russell and Whitehead never completed a fourth volume and parted company soon after. But the worse (or best) was yet to come. Perhaps we could go back to the original approach and list and check well formed formulae?

Turning-Chaitin Uncomputability

Incompleteness is only part of the problem. Recall the approach suggested by Hilbert's challenge in terms of the analogy of the algorithmic program as stated by

Chaitin. Alan Turing in 1934 stated this exact same problem in order to address some bothersome problems in number theory. Turning asked the question, “Can we truly show that arithmetic is both consistent and complete?” But he asked in the question using the same formulation from Hilbert’s 1900 problem: an algorithmic search for **information** about the well formed formulae. The requirement of consistency and completeness is approachable by what Hilbert called the Entscheidungproblem or the decision problem.

> Solving the decision problem for a formal axiomatic system is giving an algorithm that enables you to decide whether any given meaningful assertion is a theorem or not. A solution to the problem is called a decision procedure. (LOC 3-4)

Hilbert’s program would require a result in the decision problem, and this result is **information**. In other words, if you put the algorithm to the test and you ran through all the well formed formulae, a decision on a statement’s provability necessarily results. You learn if the well formed formula is a theorem. In other words, you gain information about the well formed formula. Gödel assumed that the system was consistent, but incomplete. With this decision algorithm, we can assume a consistent system that is incomplete, but still attempt to use the decision algorithm to check for theorems. Even after Gödel there was some hope to that methods and work of Russell and Whitehead would have limited success. In principle this could work but, as we saw before, it does not seem very practical given the number of possible statements.

For the point of view of the computer age, such an idea might be reasonable. So it is not surprising that is was Alan Turning, one of the first designers of the computer, who gave a resolution to the issue. Unfortunately, in 1936, well before microchip computers were available, Turing showed that “there could be no decision procedure.” (LOM 6) Turing proved that one could not resolve a decision procedure for all theorems[11]. Not only is such a system incomplete, but Hilbert's scheme must fail. It is Turning's result and not Gödel's that "really is what destroys Hilbert's dream." (LOM 7)

Turing demonstrated that the resolution of a fully demonstrable logical system would hang on the idea of computability. A computable algorithm is one for which there is a method to calculate or compute the algorithm to a result, to the point where information results. The assumption for computability is that the algorithm for a particular theorem will **halts** or complete itself in a finite time of processing, i.e. the algorithmic program ends. Now consider a computer program that will list all possible algorithms, i.e. a program that will list all possible well formed formulae.[12] If the program halts the problem is computable, or, alternatively, the process is said to be denumerable or countable. Using a Cantor diagonal argument, one can check if the problem is always denumerable, i.e. the well formed formulas can always be counted.[13] The question is "Does the program work?" What happens if the program fails to print a line or fails to compute a result? What if the program fails to stop or halt?

Turning proved that there is no algorithm, no mechanical procedure, which will decide if the nth computer program ever outputs an nth result.[14] There is no guarantee the program will halt. Anyone who has experience programming recognizes this an infinite looping problem, like trying to get simple division algorithm to divide by zero. The program gets caught up within the algorithm and can't escape the loop. Such a programming situation is said to be uncomputable.

Randomness

Recently, Chaitin has used contemporary algorithmic coding and processing and he that the algorithm to find and check all well formed formulae in the logic of arithmetic will fail to halt. The system of arithmetic is therefore uncomputable. Not a wholly unexpected result but disquieting for mathematicians. This demonstration is preformed by using a coded set of instructions of set theory in the programming language LISP, a low level or machine-language program similar to assembler. These programs use the chip processing and memory hardware directly within the instructions of the program and not through another interfacing program and are therefore free of external or confounding factors. Chaitin has made this code available and his demonstration is relatively reproducible.[15] As a corollary result, finding that the algorithm is uncomputable means that the system is incomplete, since if it the decision procedure fails to halt some theorems cannot be proved. Still nothing earth shattering considering.

The earth shattering part is the result of Chaitin's approach in simplifying both Gödel and Turning's procedures, leads to a demonstration of inherent **randomness** in number theory[16], the basic theory of addition and subtraction of numbers. This is a remarkable claim and is demonstrated using the concept of the halting probability. the halting probability is an expression of the probability that the algorithmic search will halt. Chaitin computed this, again using LISP, and expressed it as $\Omega = \sum_{p \ halts} 2^{-|p|}$. The characteristics of this value are important. Chaitin suggest that his halting probability is an uncomputable real between zero and one.[17] To understand this value, one must start by converting Ω into a binary string, a representation composed of only 1's and 0's. For this particular halting probability, such a representation will require as much information as the original number. If you have N bits of information for Ω, then the representation requires N bits, and you cannot reduce or compress the information contained in the expression. We say that this kind of information, the information in ? , is irreducible. In addition, if you were to examine a particular digit of ? , you would find that the probability of the digit being particularly a 1 or 2 or 3 … or 9, would be 10%. Unfortunately, this is the same probability of any of these digits being chosen randomly![18] If you flipped a coin to decide a particular digit of the binary version of ? , you would do as well as trying to compute the digit using an algorithmic program. Not good as this (LOM 16) means that the digit is purely random.

Furthermore, Chaitin then assembles a Diophantine[19] equation such that the equation "has a finite number of solutions if a particular individual bit of ? is a 0, and it has infinite number of solutions if that bit is a 1." (LOM 20) Thus, to decide if the Diophantine equation has either finite or infinite solutions is the same as trying to compute a particular digit of ? , which has characteristic randomness! The mathematics of Diophantine equations will have this randomness, a randomness found in elementary number theory.

Let us recap. Hilbert suggested a problem, that all theorems in arithmetic are provable using an axiomatic system that is consistent and complete. Gödel proved the system could be consistent but could not be complete. Turning proved Hibert's original algorithmic search program could be represented, but the system is essentially uncomputable, in addition to being incomplete. Chaitin demonstrated that the halting program can be expressed as ? , Chaitin's halting probability, a well known Borel real number (characteristically random). This means that Hilbert's search program results in a system that is uncomputable, incomplete and also contains some kind of randomness in in its basic axiomatic. Do you think Hilbert would be happy with this result?

<u>Discussion</u>

We already knew that logicism failed with Gödel. Chaitin shows that a construction in number theory is systematically incomplete, uncomputable with features of bit randomness. This well and truly means that Hilbert's problem is ended. Chaitin puts the final nail the coffin: the pursuit of an algorithmic search for Hilbert's well formed formulae leads to results so random that you might as well flip a coin to decide on any particular theorem. One might as well study the logic of presidential elections.

Not quite. The bright side is the same one realized 1931. Why did Gödel's result go virtually unnoticed, why did mathematicians, among others, generally ignore randomness in dynamical systems in the 1970s? Nagel and Newman believed that "neither Gödel's paper nor its content was intelligible to most mathematicians."[20] Incompleteness, uncountable and randomness in the halting problem shows that Hilbert's program does not work.

> The discovery that there are number-theoretical truths which cannot be demonstrated formally does not mean there are truths which are forever incapable of becoming known, or that a "mystic" intuition (radically different in kind and authority from what is generally operative in intellectual advances) must replace cogent proof. … It is an occasion, not for dejection, but for a renewed appreciation of the powers of creative reason. [21]

What Gödel, Turing and Chaitin do not show, and do not even attempt to show, is that the axiomatic method does not work. Russell and Whitehead demonstrated that given a set five axioms, along with some logical principles, thousands of pages of proof and demonstration at the most pedantic level result. This is no useless task. Set theory and number theory, direct off-shoots of Principia Mathematica, have proven vital exploring a number of fields in mathematics and science in general. Hilbert was right! He wasn't a twit after all. The program can result in useful and meaningful knowledge. How this came about certainly is not as

Hilbert predicted, but it came about none the less and I am sure that Hilbert would be satisfied with this outcome. The usefulness of mathematical proof is still as solid and valuable as it was in 1900, if not stronger. If the system is to be consistent, one cannot prove all theorems in the system. But we can certainly can and will prove important theorems.

Is mathematics a system of formal computations? This the same question as the logicist's: "Is mathematics a system of formal logic?" No, we know this is not the case. The randomness of Chaitin's work is a feature of mathematics, yet not necessarily central to the foundations of mathematics. But we new this already without going to the trouble of Diophantine equations and halting probabilities by considering non-linear iterative dynamics of simple quadratic equations[22]. Chaitin's focus on the proof structure as a computational representation of the logical structure is limited to Hilbert's problem. We know there is more to mathematics than what Russell and Whitehead had attempted and failed to complete. And, mathematics has thrived since.

Randomness as a feature of reality is the real question on my mind. It is no surprise to me that Chaitin found randomness in the structure of mathematics, although this may be a real and final blow to the few contemporary mathematicians holding out hope for logicism. But, even Chaitin admitted that he could not **increase** randomness in a numerical string and this is what evolution in nature needs. For evolution to even be possible, randomness must be a feature of nature's structure, not a feature of mathematical theory of numbers, not a feature of the search for a halting program but a feature of information.

> "Countervailing the general tendency of the universe toward increased entropy, as specified by the second law of thermodynamics, is the order and decreased entropy produced by complex systems."[23]

Jungerman is talking about the complex system of evolution on earth. Randomness must be a characteristic of reality, otherwise the order that results from evolution would not be possible according to thermodynamic theory. Chaitin has failed to find this randomness using his program size complexity.[24] This failure denies evolution, or is at least a major problem. Nevertheless, Chaitin's work is very important since he shows that the next step is to find randomness in the structure of reality. This randomness is likely to involve similar algorithmic information as found in Chaitin's problem and it is this concentration on algorithmic problems, on processes of information contained in a string, which holds promise for an advancement in understanding of our reality. This is not an unlikely result since we can easily find exactly this kind of information in a predator-prey model or even within the DNA structure itself.

> Consciousness does not seem to be material, and information is certainly immaterial, so perhaps consciousness, and even the soul, is sculpted in information, not matter. … The conventional view is that matter is primary, and that information , if it exists, emerges from matter. But what if information is primary, and matter is the secondary phenomenon![25]

This really upsets the apple cart, or should I say the apple "Des–**cart-**es" ! If one would postulate information as THE primary quality, and if it has this Chaitin randomness to it, then we should find that its algorithmic processes can have chaotic behavior, but we can then truly understand. The biggest obstacle to this is the very fabric of traditional philosophy – Aristotelian metaphysics.

---

[1] The first computer was named Colossus (1943), not to be confused with the 1966 Eniac computer. Colossus was a secret enterprise of the World War II western coalition, located at Bletchley Park ('*Station X*'). Image from http://en.wikipedia.org/wiki/Colossus_computer

[2] David Hilbert, "Mathematical Problems", lecture delivered before the International Congress of Mathematicians at Paris in 1900, translated by Mary Winton Newson for Bulletin of the American Mathematical Society 8 (1902) 437-479. I am quoting from a text version of the paper and page numbering will vary slightly. p. 439

[3] Hilbert, p. 443

[4] Ernest Nagel and James R. Newman, *Gödel's Proof*, revised edition, (New York University Press: New York, 2001) p. 25

[5] F_o ✍ P, is really F ✍ (T v F) ? T , i.e. true for P or not P.

---

[6] Gregory Chaitin, “The Decline and Fall of Reductionism in Pure Mathematics”, in *The Limits of Mathematics*, (Springer-Verlag: Singapore, 1998) Future references to this text will be cited in the text as LOM

[7] Henri Poincare was famous for them, especially for his diagrams and calculations of chaos, although the meanings of his results only became apparent in the 1970s.

[8] The school of thought that assumes that one could reduce all of mathematics to logic is referred to, alternatively, as positivism, logical positivism or logicism. See Marc Corbeil, Mathematics and Logic, (Concordia University Archives: Montréal, 1997) This paper is available at the Concordia University archives, Library or at www.mcorbeil.com.

[9] Bertrand Russell and Alfred North Whitehead, Principia Mathematica, (Cambridge Press, Cambridge, 1989) Using Peano’s Axioms Russell and Whitehead take almost 70 pages just to get 1+1 = 2 ! A forth volume on geometry was planned but never completed.

[10] Kurt Gödel, *Umber formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme [On Formally Undecidable Propositions of Principia Mathematica and Related Systems – 1931],* translated by B. Meltzer, (Dover Publications: 1992)

[11] Alan M. Turing, "*On Computable Numbers with an Application to the Entscheidungsproblem*," Proc. London Math. Soc., Vol. 42, pp. 230-265, 1936

[12] The machine that does this is called a Turning Machine and s the inspiration that lead to the first real computer in 1943, called Colossus and then the modern computer.

[13] Although possibility infinite. By countable here we mean that a one-to-one correspondence with the set of natural numbers is possible. The set of rational numbers, all the natural numbers and fractions, anything that is writable as p/q, where p, q are both integers and g not zero; this set is countable by Cantor’s argument. (Diagonalization Theorem). The real numbers is the set of rational number union with the set of irrational numbers like Pi, square root of 2, non-repeating, non-terminating decimal representations. The real numbers fail the Diagonalization Theorem and are not countable. Thus, there are at least two types of numerical infinity: countable and uncountable. Finite sets, by the way, are countable by definition.

[14] Chaitin actually describes this in terms of as a numerical process and translates the algorithmic issue as a numerical process. The output is the nth digit.

[15] I lack the expertise in information theory required to compress and fully qualify the results but I have yet to find dissenters in the literature, which, is as one imagines, rather small. Steven Wolfram, an “super expert” programmer and creator of Mathematica, a computer algebra system used by professional mathematicians, agrees with Wolframs result. If anyone has the expertise in this field, Wolfram is it.

[16] This is worse than randomness in arithmetic.

[18] This follows since ? is a Borel normal number. See LOM p. 14-5

[19] This is making use of Hilbert’s tenth question involving Diophantine equations in elementary number theory. The equation ax + by = c is a simple linear Diophantine equation. Given a, b, c one can solve this equation for a particular solution or a general solution (x,y).

[20] Nagel and Newman, Ibid., p. 1

[21] Nagel and Newman, Ibid., p. 113-4

[22] Reference self later or a paper on logistic equation.

[23] John A. Jungerman, *World in Process: Creativity and Interconnection in the New Physics*, (SUNY, New York, 2000) p. 135

[24] Gregory Chaitin, *The Unknowable*, (Springer-Verlag: Singapore, 1998), p. 108

[25] Gregory Chaitin, *Unknowable*, Ibid., p. 106